Math 206B Lecture 21 Notes

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1 The Jacobi-Trudi Identity

1.1 Connection to Frobenius' theorem

Theorem 1.1 (Jacobi-Trudy[1]). *Let* $\lambda = (\lambda_1, \dots, \lambda_\ell)$. *Then*

$$s_\lambda = \det(h_{\lambda_{1-i+j}})_{i,j=1}^{\ell}.$$

This should remind you of the following theorem:

Theorem 1.2 (Frobenius).

$$\chi^\lambda = \sum_{\omega \in S_\ell} \operatorname{sgn}(\omega) \zeta^{\lambda + \omega\rho - \rho},$$

where ζ^μ *is the character of* M^μ *and* $\rho(\ell - 1, \ell - 2, \dots, 1, 0)$.

In fact, these are the same.

Example 1.1. Let $\lambda = (4, 2, 1)$, so $\ell = 3$. Then

$$s_{(4,3,2)} = \begin{vmatrix} h_4 & h_5 & h_6 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix} = h_{(4,2,1)} - h_{(5,1,1)} - h_{(4,3)} + h_{(6,1)}$$

The Frobenius formula in this case says

$$\begin{aligned} \chi^{(4,2,1)} &= \zeta^{(4,2,1)-(2,1,0)+(2,1,0)} - \zeta^{(4,2,1)-(1,2,0)+(2,1,0)} \\ &\quad - \zeta^{(4,2,1)-(2,0,1)+(2,1,0)} + \zeta^{(4,2,1)-(0,2,1)+(2,1,0)} + 0 + 0 \\ &= \zeta^{(4,2,1)} - \zeta^{(5,1,1)} - \zeta^{(4,3)} + \zeta^{(6,1)}. \end{aligned}$$

[1] You can find this in section 7.16 of Richard Stanley's Enumerative Combinatorics volume 2.

1.2 Proof of the identity

The idea of the proof is term cancellation. We expand the determinant and show that a lot of terms cancel. Here is an analogy.

Proposition 1.1. *Let* $A = (a_{i,j})$, $B = (b_{i,j})$ *be* $n \times n$ *matrices. Then* $\det(AB) = \det(A)\det(B)$.

You prove this by representing both sides as counting something and showing that the extra terms in $\det(AB)$ compared to $\det(A)\det(B)$ cancel out in pairs.

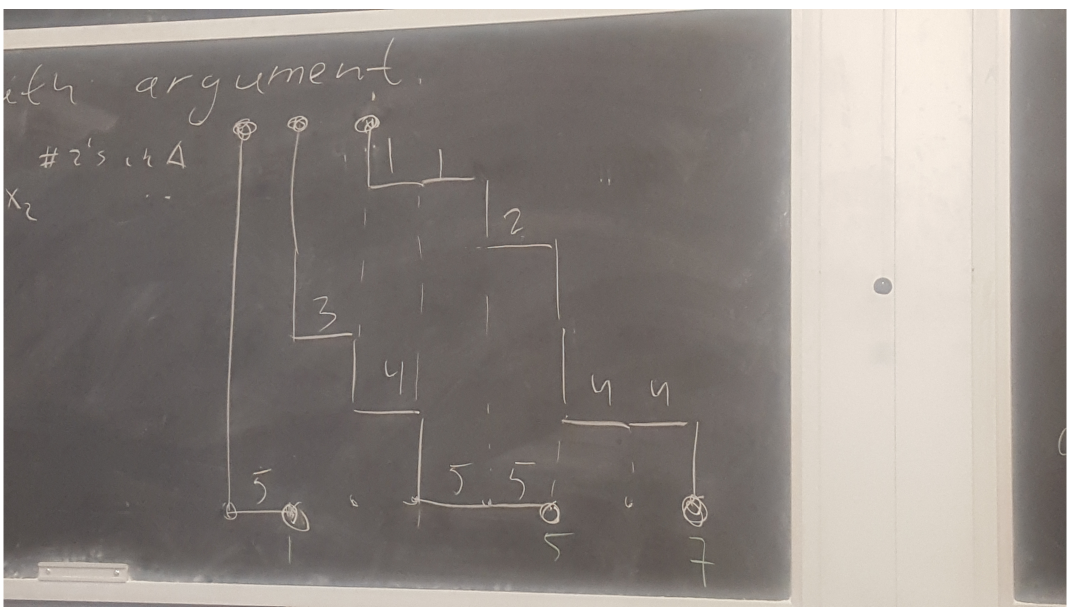
Proof. The idea is a non-crossing path argument, like you can use to prove the statement about determinants.

$$S_\lambda = \sum_{A\,\mathrm{SSYT}(\lambda)} x^A = \sum_A x_1^{\#1\mathrm{s}} x_2^{\#2\mathrm{s}} \cdots$$

Take A and construct a system of paths. For example, if

$$A = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 4 \\ \hline 3 & 4 & 5 & 5 \\ \cline{1-4} 5 \\ \cline{1-1}\end{array}$$

construct the system of paths



Fix the starting points and ending points. Now consider all systems of paths from the starting points to the ending points. A path Q will produce the monomial $x^Q =$

$x_1^{\#1\text{s}} x_2^{\#2\text{s}} \cdots$, and if we switch two paths that intersect, we get a $\mathrm{sgn}(\sigma)$ coefficient in front. Then each system with a pair of paths that intersect somewhere will not be counted because we can just switch the paths after the intersection point; in this case, the contribution of the paths to the sum will cancel. Since the paths define polynomials h_λ with this definition, we are done. $\square$